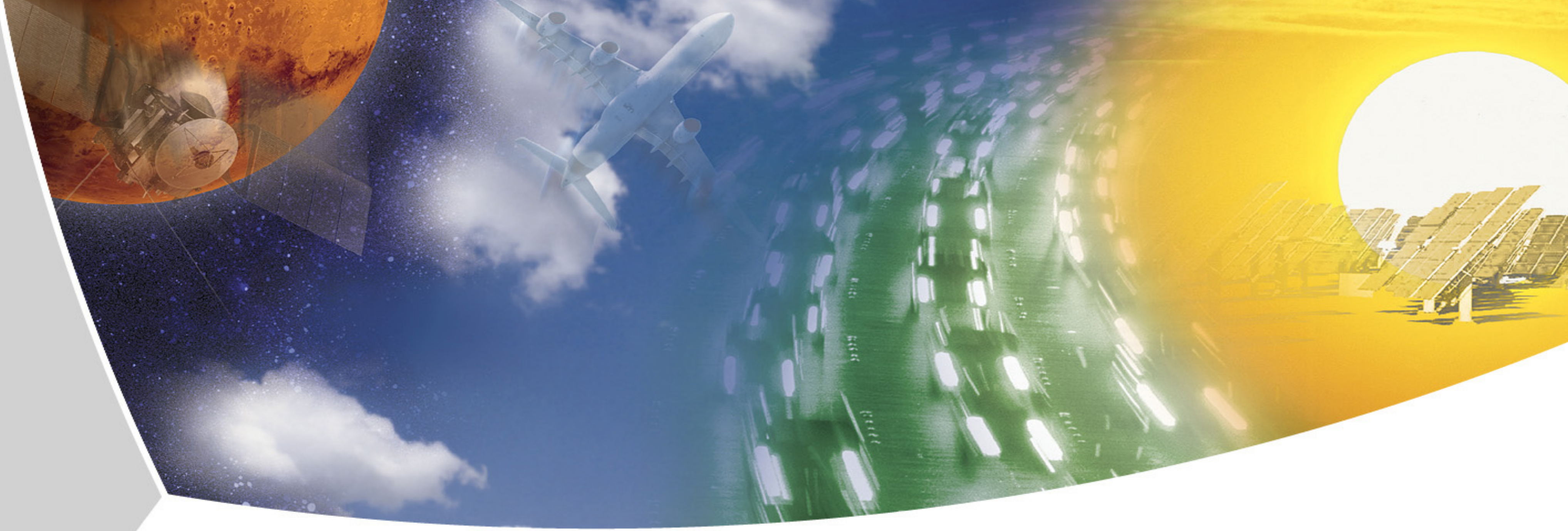

Error estimation and adjoint-based refinement for multiple force coefficients in aerodynamic flow simulations

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Motivation

Given N target quantities, e.g. following aerodynamic force coefficients

- the pressure induced force coefficients: cdp, clp, cmp
- the viscous force coefficients: cdf, clf, cmf

i.e. 6 force coefficients in 2d or 10 force coefficients in 3d.

The current approach: Error estimation and adjoint-based refinement requires the solution of N adjoint problems.

Goal: Replace the N adjoint problems by two auxiliary problems (1 adjoint problem and 1 adjoint adjoint problem) irrespective of the number of target quantities.

Error estimation for single target quantities

Given a discretization: find $\mathrm{u}_h \in \mathrm{V}_h$ such that

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{1}$$

and a target quantity J.

Computed: $J(\mathrm{u}_h)$, exact (but unknown): $J(\mathrm{u})$, what is $J(\mathrm{u}) - J(\mathrm{u}_h)$?!

Using a duality argument we obtain an error representation wrt. $J(\cdot)$:

$$\begin{aligned} J(\mathbf{u}) - J(\mathbf{u}_h) &= \mathcal{R}(\mathbf{u}_h, \mathbf{z}) := -\mathcal{N}(\mathbf{u}_h, \mathbf{z}) \\ &\approx \mathcal{R}(\mathbf{u}_h, \tilde{\mathbf{z}}_h) = \sum_\kappa \eta_\kappa. \end{aligned}$$

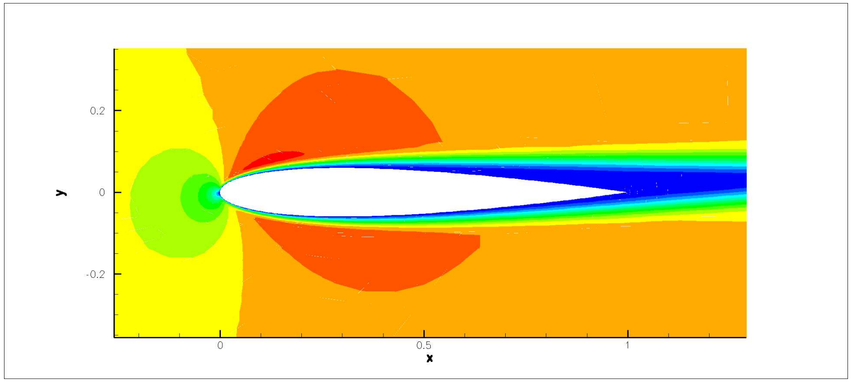
where $\tilde{\mathrm{z}}_h$ is the solution to the discrete adjoint problem: find $\tilde{\mathrm{z}}_h \in \tilde{\mathrm{V}}_h$ such that

$$\mathcal{N}'[\mathbf{u}_h](\mathbf{w}_h, \tilde{\mathbf{z}}_h) = J'[\mathbf{u}_h](\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h,$$

and η_κ are adjoint-based indicators which are particularly suited for the accurate and efficient approximation of the target quantity $J(\mathrm{u})$.

Example: ADIGMA MTC3 test case

Laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$ around NACA0012 airfoil:



We are interested in the

1. **pressure induced drag:** $J(\mathrm{u}) = c_{dp}$
2. **viscous drag:** $J(\mathrm{u}) = c_{df}$
3. **total lift:** $J(\mathrm{u}) = c_l$
4. **total momentum:** $J(\mathrm{u}) = c_m$

Error estimation for single target quantity: $J(\mathbf{u}) = c_{dp}$

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(\mathbf{u}) = c_{dp}$ **(pressure induced drag), Ref.value:** $J^{ref}_{cdp}(\mathbf{u}) = 0.02380$

		error in c_{dp}		
cells	DoFs	exact	estimate	ratio
400	6400	1.034e-03	-1.404e-03	-1.36
652	10432	3.341e-03	2.959e-03	0.89
1090	17440	4.045e-04	5.712e-04	1.41
1801	28816	-2.079e-04	-1.091e-04	0.52
3034	48544	-2.344e-04	-1.890e-04	0.81
5047	80752	-1.529e-04	-1.387e-04	0.91
8527	136432	-8.055e-05	-7.536e-05	0.94
14410	230560	-4.357e-05	-3.762e-05	0.86
24406	390496	-2.366e-05	-2.314e-05	0.98

Error estimation for single target quantity: $J(\mathbf{u}) = c_{df}$

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(\mathbf{u}) = c_{df}$ **(viscous drag), Ref.value:** $J_{cdf}^{ref}(\mathbf{u}) = 0.0322835$

		error in c_{df}		
cells	DoFs	exact	estimate	ratio
400	6400	1.076e-02	1.525e-02	1.42
655	10480	-2.973e-03	-2.592e-03	0.87
1093	17488	-1.415e-03	-1.418e-03	1.00
1804	28864	-3.947e-04	-4.326e-04	1.10
2989	47824	-9.136e-05	-1.116e-04	1.22
5110	81760	-3.787e-05	-4.518e-05	1.19
8476	135616	-1.919e-05	-2.071e-05	1.08
14185	226960	-1.319e-05	-1.619e-05	1.23
23638	378208	-1.048e-05	-1.052e-05	1.00

Error estimation for single target quantity: $J(\mathbf{u}) = c_l$

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(\mathbf{u}) = c_l$ **(total lift), Ref.value:** $J_{cl}^{ref}(\mathbf{u}) = 0.037286$

		error in c_l		
cells	DoFs	exact	estimate	ratio
400	6400	-1.175e-01	-5.867e-02	0.50
658	10528	6.548e-03	6.841e-03	1.04
1108	17728	-1.292e-03	-1.159e-03	0.90
1861	29776	-1.784e-03	-1.891e-03	1.06
3118	49888	-1.239e-03	-1.266e-03	1.02
5236	83776	-6.504e-04	-6.704e-04	1.03
8746	139936	-2.623e-04	-2.622e-04	1.00

Error estimation for single target quantity: $J(\mathbf{u}) = c_m$

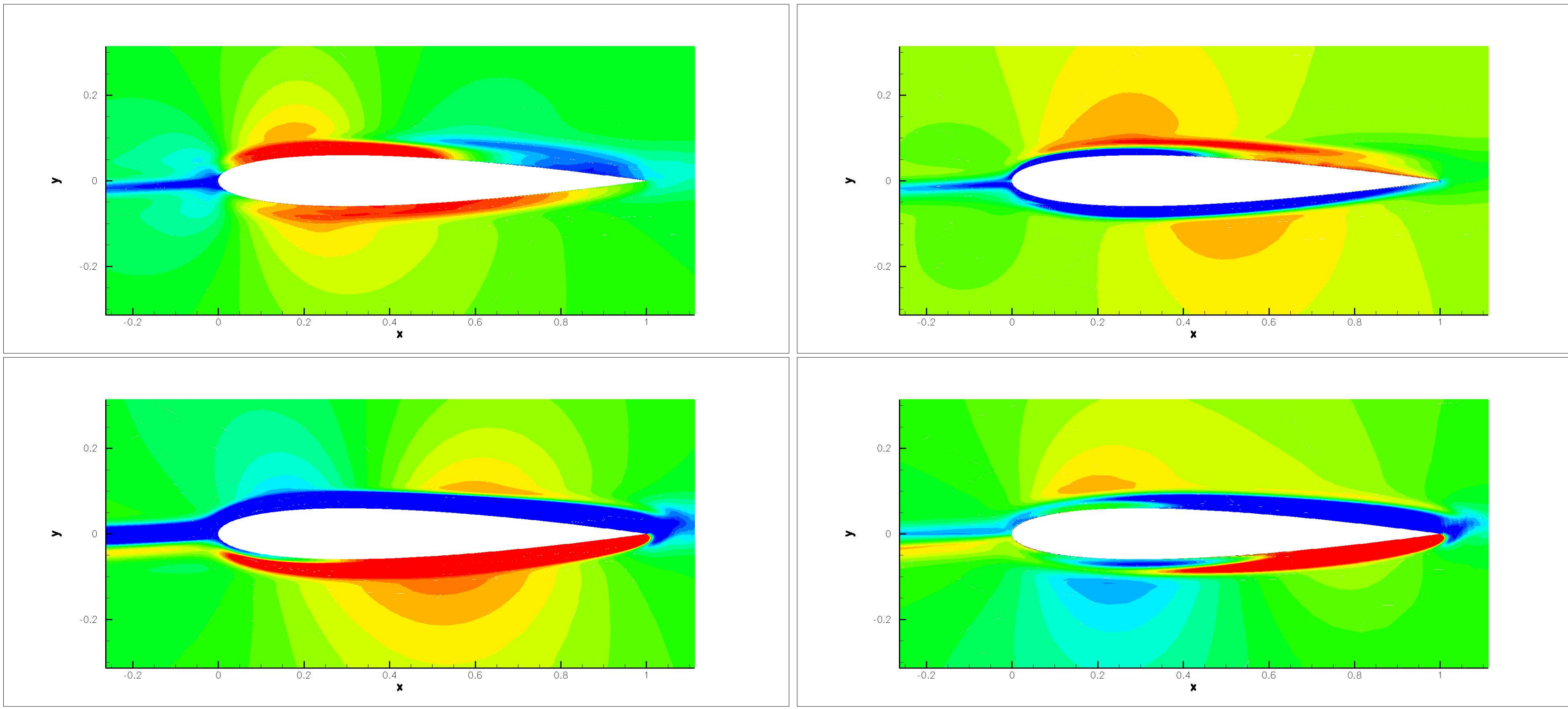
Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(\mathbf{u}) = c_m$ **(total moment), Ref.value:** $J_{cm}^{ref}(\mathbf{u}) = -0.01661$

		error in c_m		
cells	DoFs	exact	estimate	ratio
400	6400	-1.221e-03	-3.035e-03	2.49
667	10672	2.883e-03	3.001e-03	1.04
1138	18208	3.862e-04	4.378e-04	1.13
1867	29872	9.083e-05	8.543e-05	0.94
3130	50080	6.199e-05	5.807e-05	0.94

Error estimation for single target quantities

z_1 **components of adjoint solutions.** **Top: cdp, cdf;** **Bottom: cl, cm.**



Error estimation for multiple target quantities

For N target quantities $J_i(\mathbf{u}), i = 1, \ldots, N$: Instead of computing N adjoint solutions

$$\mathcal{N}'[\mathbf{u}_h](\mathbf{w}_h, \tilde{\mathbf{z}}_{i,h}) = J_i'[\mathbf{u}_h](\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h, \qquad i = 1, \ldots, N,$$

we now compute *one* adjoint-adjoint problem: find $\tilde{\mathrm{e}}_h \in \tilde{\mathbf{V}}_h$ such that

$$\mathcal{N}'[\mathbf{u}_h](\tilde{\mathbf{e}}_h, \mathbf{w}_h) = \mathcal{R}(\mathbf{u}_h, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h,$$

to obtain error estimates for the N target quantities

$$J_i(\mathbf{u}) - J_i(\mathbf{u}_h) \approx J_i'[\mathbf{u}_h](\mathbf{e}) \approx J_i'[\mathbf{u}_h](\tilde{\mathbf{e}}_h), \qquad i = 1, \ldots, N.$$

Additionally, we compute *one* adjoint problem: find $\tilde{\mathbf{z}}_{c,h} \in \tilde{\mathbf{V}}_h$ such that

$$\mathcal{N}'[\mathbf{u}_h](\mathbf{w}_h, \tilde{\mathbf{z}}_{c,h}) = J_c'[\mathbf{u}_h](\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h,$$

for a combined target functional $J_c(\mathbf{u})$ to obtain the adjoint-based indicators

$$J_c(\mathbf{u}) - J_c(\mathbf{u}_h) \approx \mathcal{R}(\mathbf{u}_h, \tilde{\mathbf{z}}_{c,h}) = \sum_\kappa \eta_\kappa.$$

which are suited to reduce e.g. the sum of absolute or relative errors of $J_i(\mathbf{u}_h)$.

Error estimation for multiple target quantities

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2°, Re = 5000$

On each mesh compute primal solution u_h and adjoint-adjoint solution $\tilde{\mathrm{e}}_h$.

Evaluate *exact* error: $J_i^{\text{ref}}(\mathbf{u}) - J_i(\mathbf{u}_h), \quad i = 1, \ldots, N,$

Evaluate error *estimate*: $J_i'[\mathrm{u}_h](\tilde{\mathrm{e}}_h), \quad i = 1, \ldots, N,$

	error in cdp		error in cdf		error in cl		error in cm	
#cells	exact	estimate	exact	estimate	exact	estimate	exact	estimate
400	1.03e-03	-2.92e-03	1.08e-02	1.62e-02	-1.18e-01	-6.59e-02	-1.22e-03	-4.36e-03
655	1.39e-03	1.38e-03	-3.02e-03	-2.89e-03	6.30e-03	4.15e-03	2.99e-03	2.67e-03
1111	-1.04e-04	8.65e-05	-1.42e-03	-1.89e-03	-8.30e-04	-6.54e-04	4.76e-04	5.11e-04
1843	-6.28e-04	-5.28e-04	-5.20e-04	-6.46e-04	-1.83e-03	-1.91e-03	6.25e-05	3.49e-05
3061	-3.96e-04	-3.51e-04	-1.61e-04	-2.25e-04	-7.34e-04	-7.69e-04	3.15e-05	3.67e-05
5146	-1.82e-04	-1.63e-04	-9.03e-05	-1.11e-04	-4.86e-04	-3.94e-04	1.09e-05	1.35e-05

Adaptive refinement for multiple target functionals (2)

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: sum of relative errors

# cells	# DoFs	exact	estimate	ratio
400	6400	3.602e+00	1.362e+00	0.38
655	10480	5.009e-01	5.036e-01	1.01
1111	17776	9.940e-02	9.135e-02	0.92
1843	29488	9.535e-02	8.884e-02	0.93
3061	48976	4.320e-02	4.299e-02	1.00
5146	82336	2.414e-02	2.537e-02	1.05

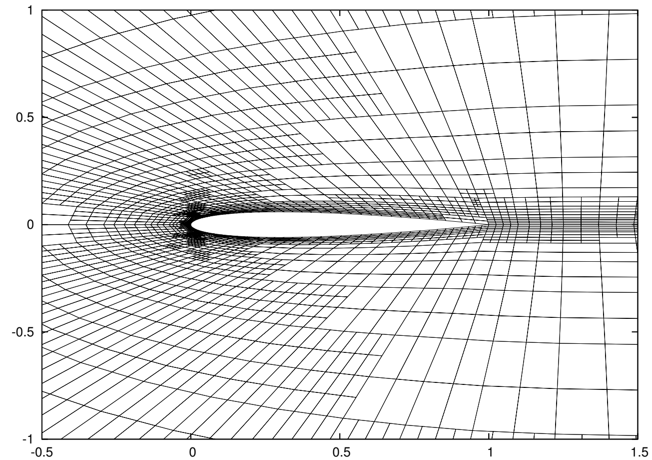
On finest mesh sum of relative errors is 2.4%. Error estimation tells us: 2.5%

Goal-oriented refinement for multiple target quantities

Example: ADIGMA MTC-3, laminar, $M = 0.5, \alpha = 2^\circ, Re = 5000$

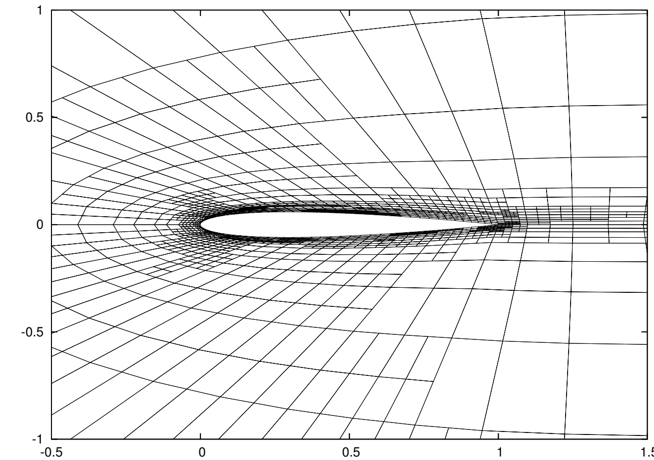
Goal: Accurate and efficient approximation of cdp, cdf, cl, cm

accuracy requirements (ADIGMA): cdp, cdf, cm: $|$error$|$ $<$5e-4, cl: $|$error$|$ $<$5e-3



residual-based refinement

8896 cells, 149.4s



adjoint-based refinement

1894 cells, 80.8s (incl. error est.)

stronger accuracy requirements: cdp, cdf, cm: $|$error$|$ $<$1e-4, cl: $|$error$|$ $<$1e-3

67660 cells, 2691.1s

8539 cells, 664.6s (incl. error est.)

The residual-based indicators

Using the error representation

$$J(\mathbf{u}) - J(\mathbf{u}_h) = \mathcal{R}(\mathbf{u}_h, \mathbf{z}) = -\mathcal{N}(\mathbf{u}_h, \mathbf{z}) = -\mathcal{N}(\mathbf{u}_h, \mathbf{z} - \mathbf{z}_h)$$

and assuming $\mathbf{z} \in [H^1(\kappa)]^5$ with $\|\mathbf{z}\|_{[H^1(\kappa)]^5} \leq C_{stab}$ we obtain

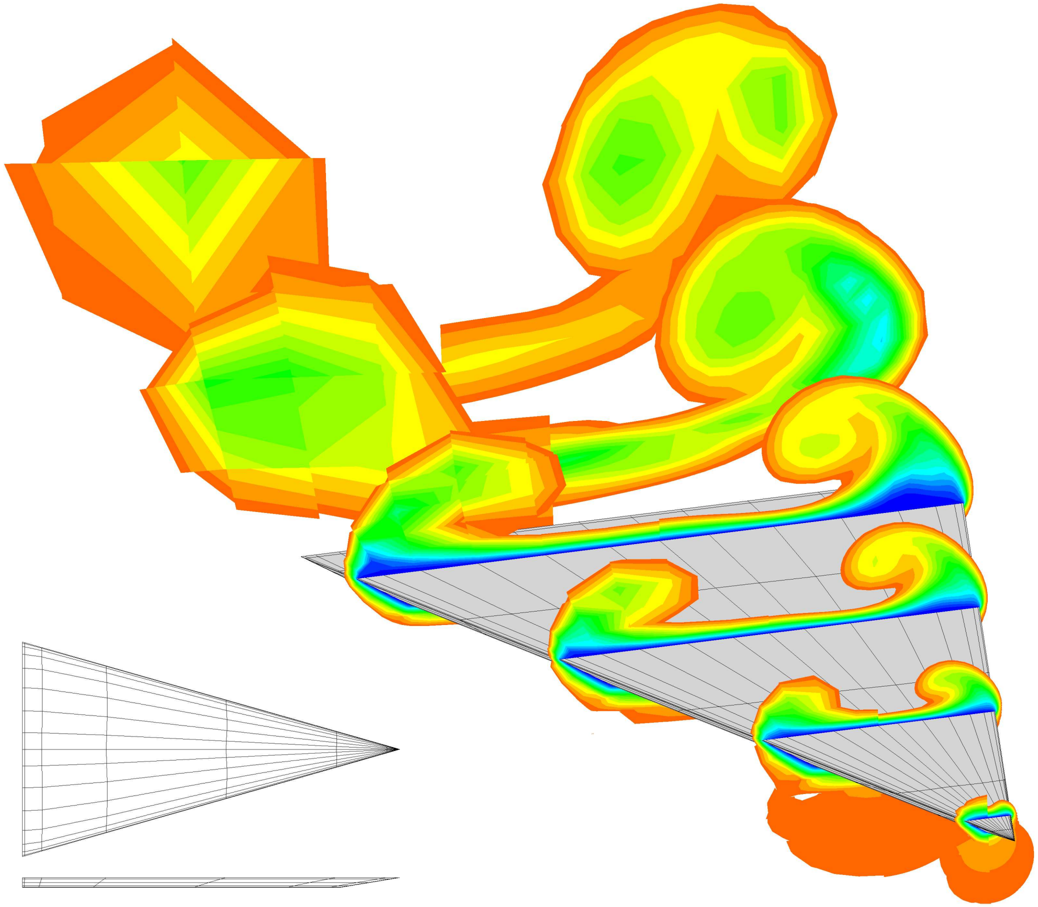
$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \leq \left(\sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^{(\mathrm{res})} \right)^2 \right)^{1/2},$$

where the residual-based indicators $\eta_\kappa^{(\mathrm{res})}$, $\kappa \in \mathcal{T}_h$, are given by

$$\eta_\kappa^{(\mathrm{res})} = h_\kappa \|\mathbf{R}(\mathbf{u}_h)\|_\kappa + h_\kappa^{1/2} \|\mathbf{r}_{\partial\kappa}(\mathbf{u}_h)\|_{\partial\kappa} + h_\kappa^{-1/2} \|\underline{\rho}_{\partial\kappa}(\mathbf{u}_h)\|_{\partial\kappa},$$

Note: $\eta_\kappa^{(\mathrm{res})}$ is independent of target quantity $J(\mathbf{u})$. Mesh refinement based on the residual-based indicators $\eta_\kappa^{(\mathrm{res})}$ targets at resolving *all* flow features.

Example: Laminar delta wing (BTC3)

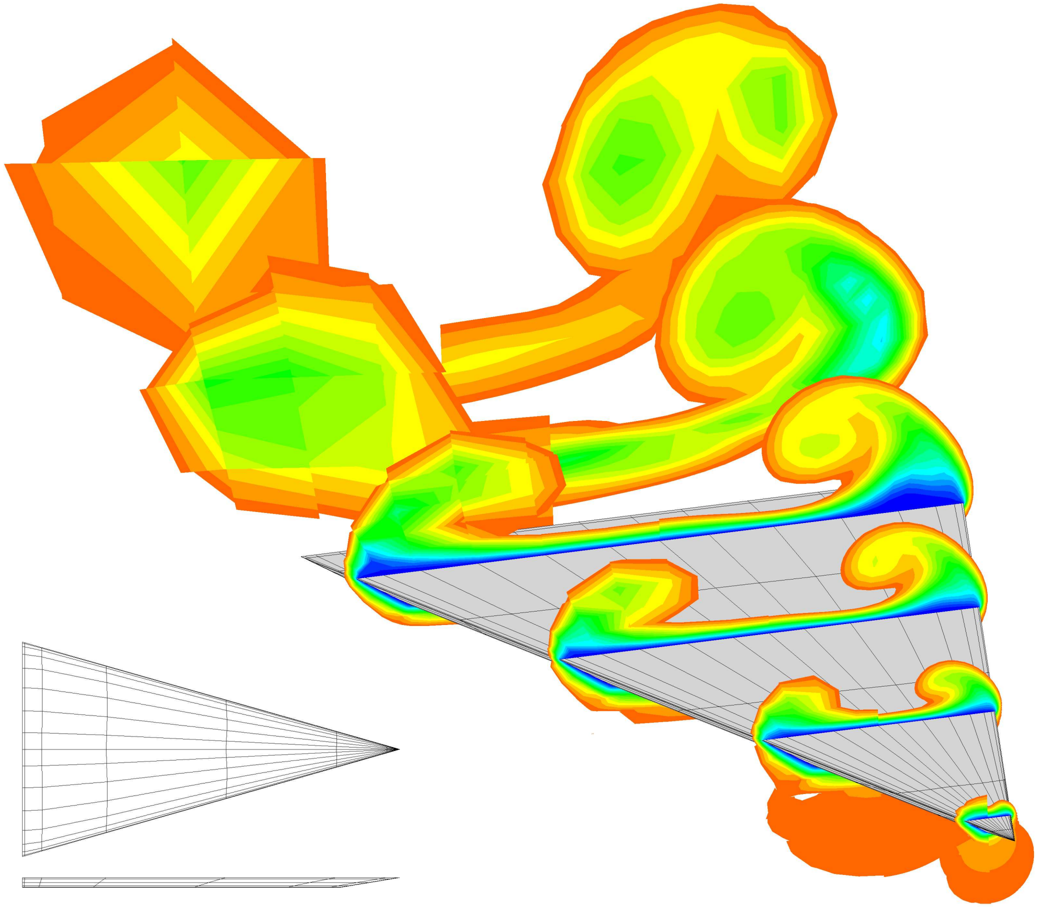


$M = 0.3, \alpha = 12.5°$,
$Re = 4000$

3264 elements for the half model

left: DG(1), 2nd order
right: DG(4), 5th order

Example: Laminar delta wing (BTC3)



$M = 0.3, \alpha = 12.5^\circ$,
$Re = 4000$

3264 elements for the half model

left: DG(1), 2nd order
right: DG(4), 5th order

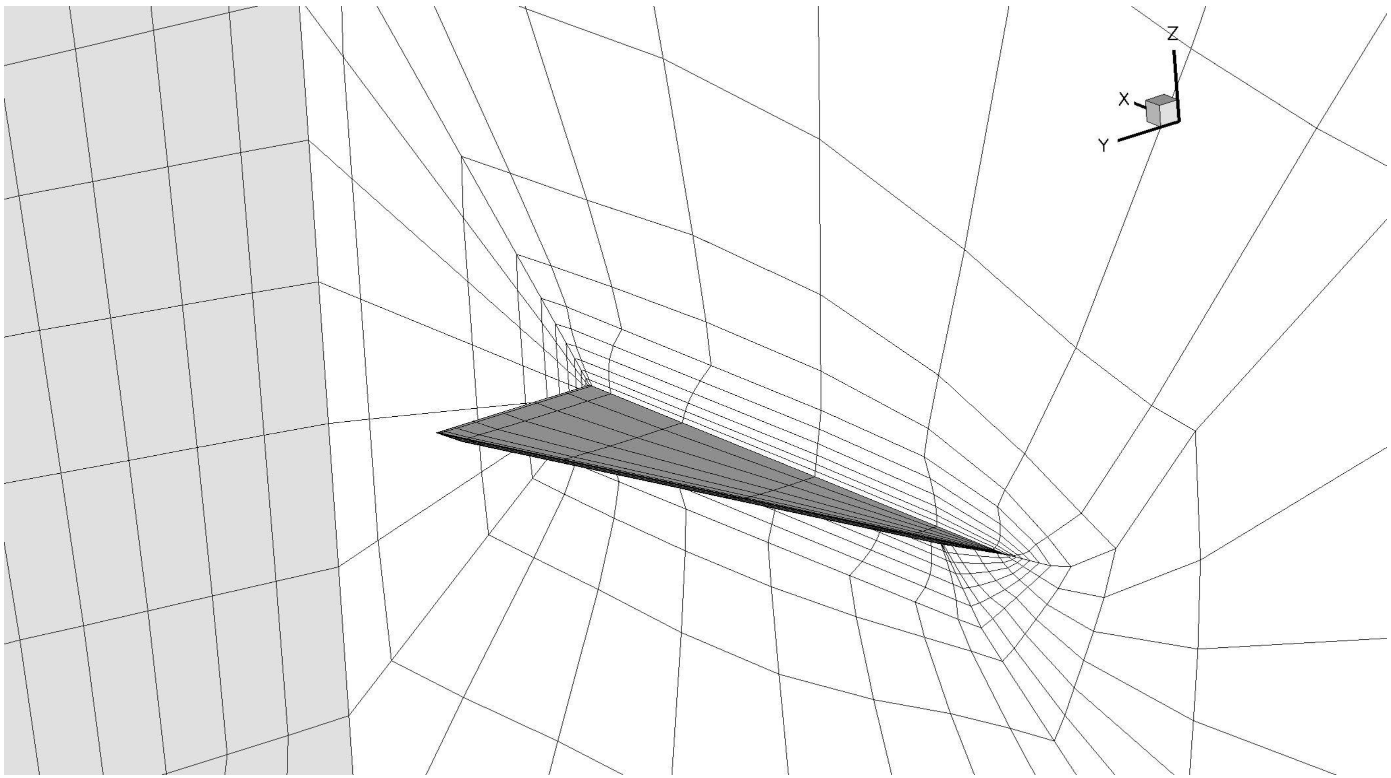
DG(1), 40 DoFs/element:
130 560 DoFs

DG(4), 625 DoFs/element:
2 040 000 DoFs

Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3$, $\alpha = 12.5°$, $Re = 4000$.

Residual-based refinement

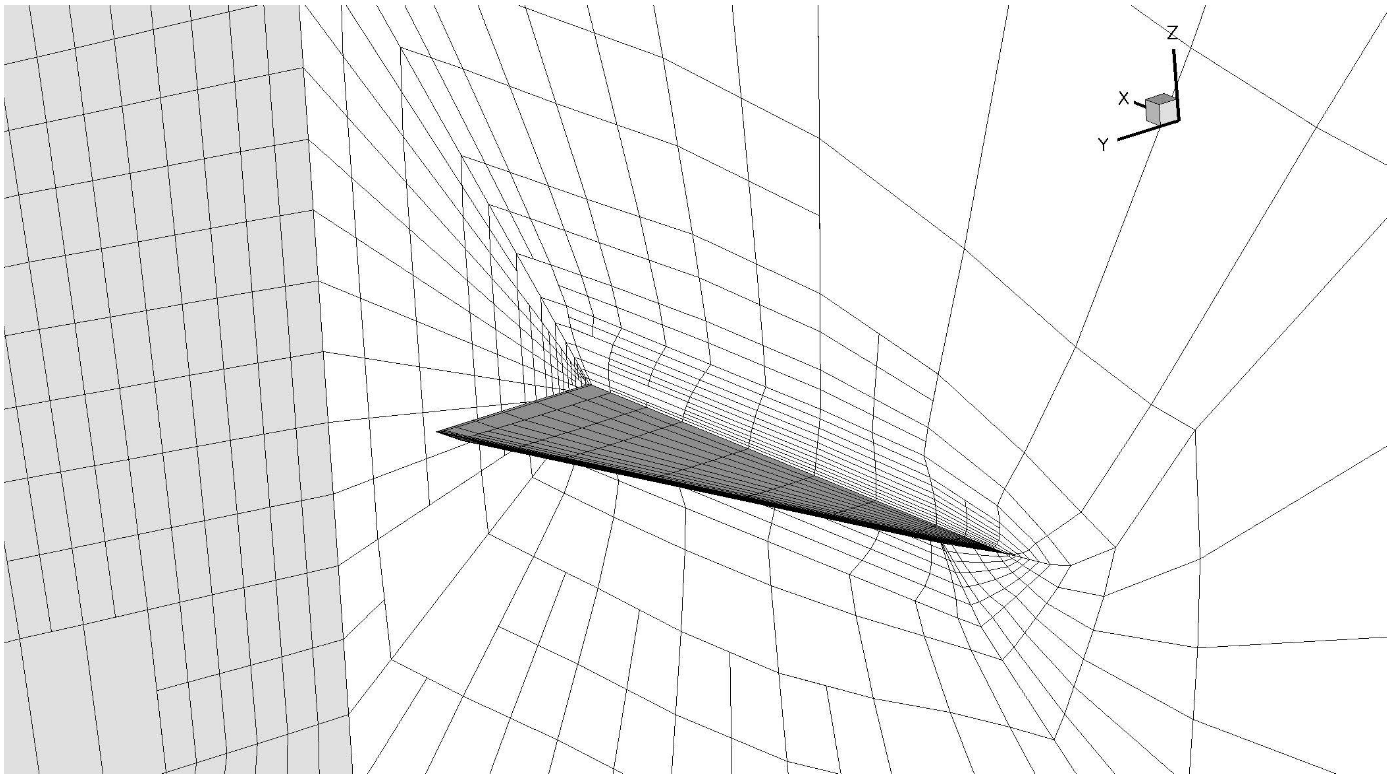


Coarse mesh: 3 264 elements, 130 560 DoFs

Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3$, $\alpha = 12.5^\circ$, $Re = 4000$.

Residual-based refinement

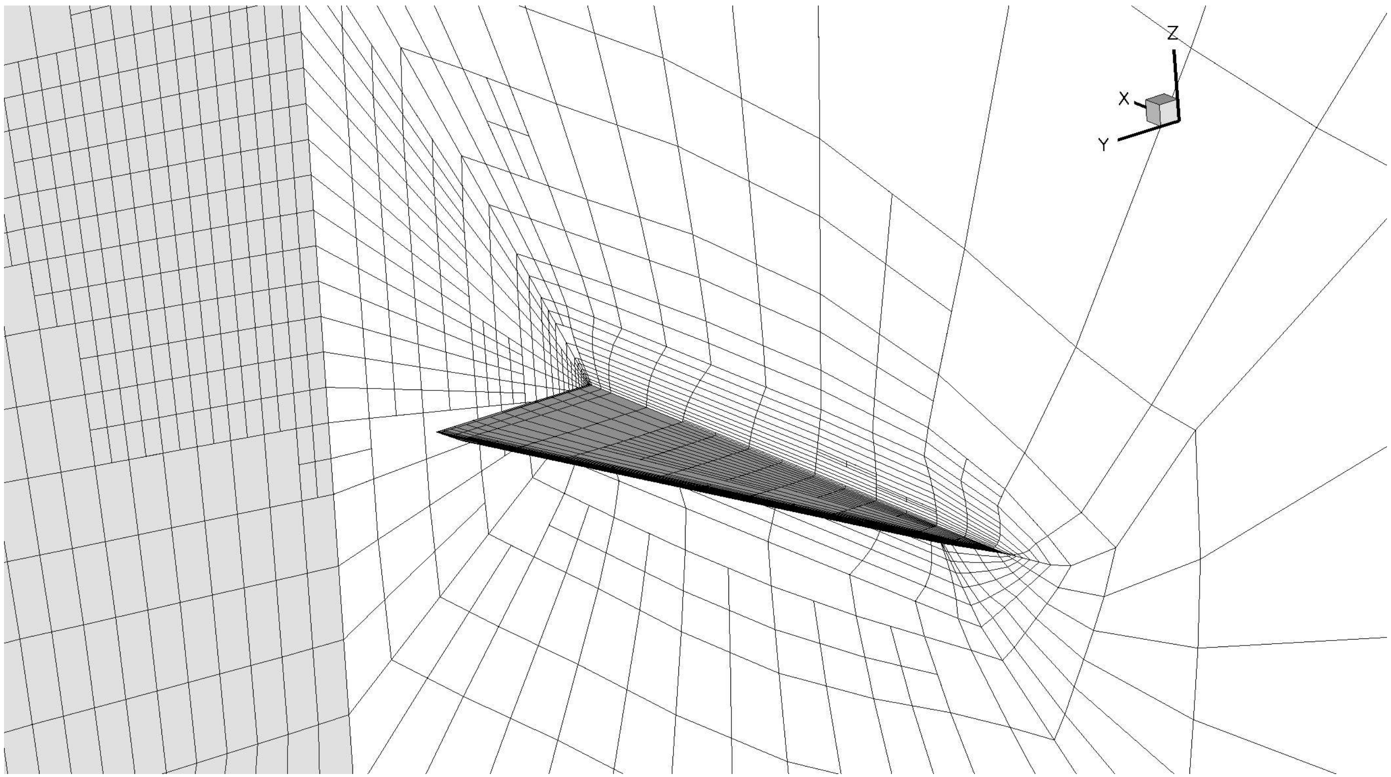


1. refinement step: 8 192 elements, 327 680 DoFs

Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3$, $\alpha = 12.5°$, $Re = 4000$.

Residual-based refinement

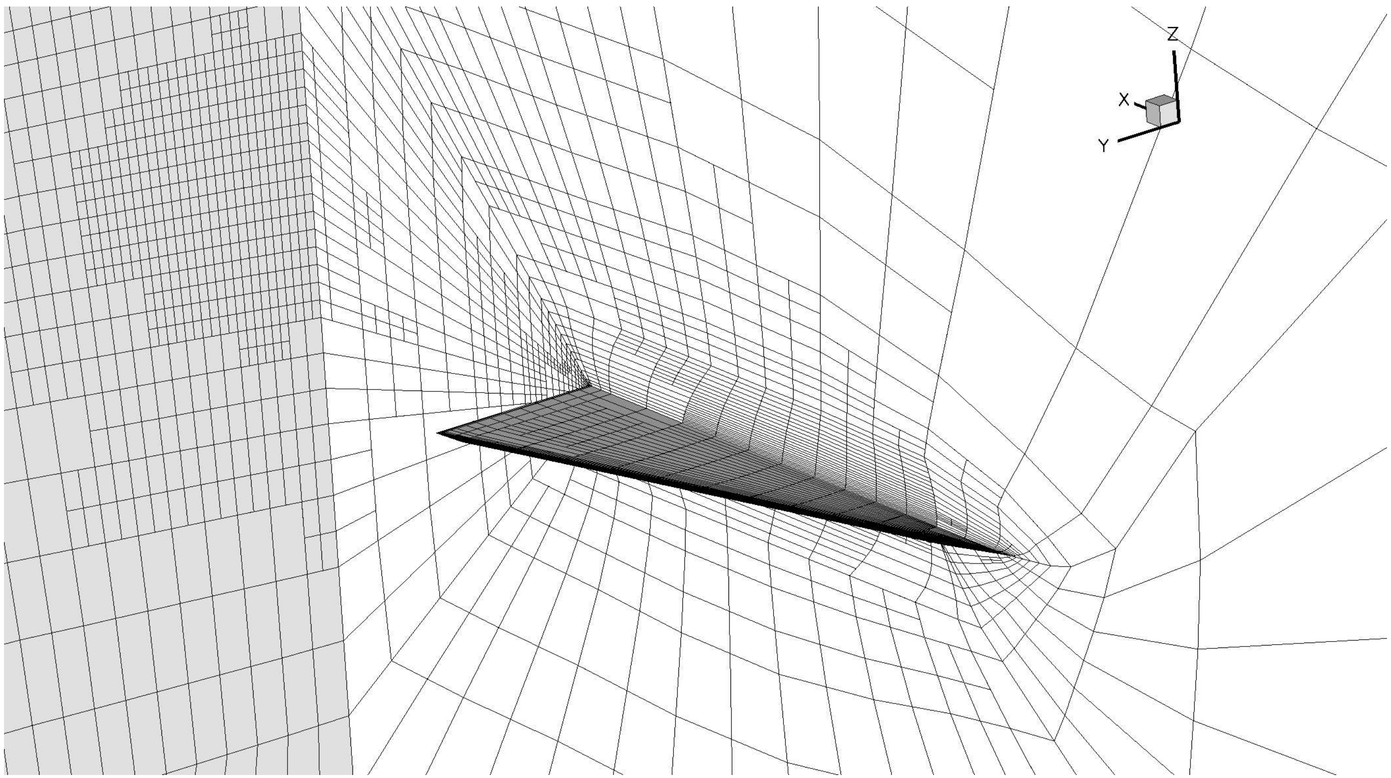


2. refinement step: 21 352 elements, 854 080 DoFs

Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3$, $\alpha = 12.5°$, $Re = 4000$.

Residual-based refinement

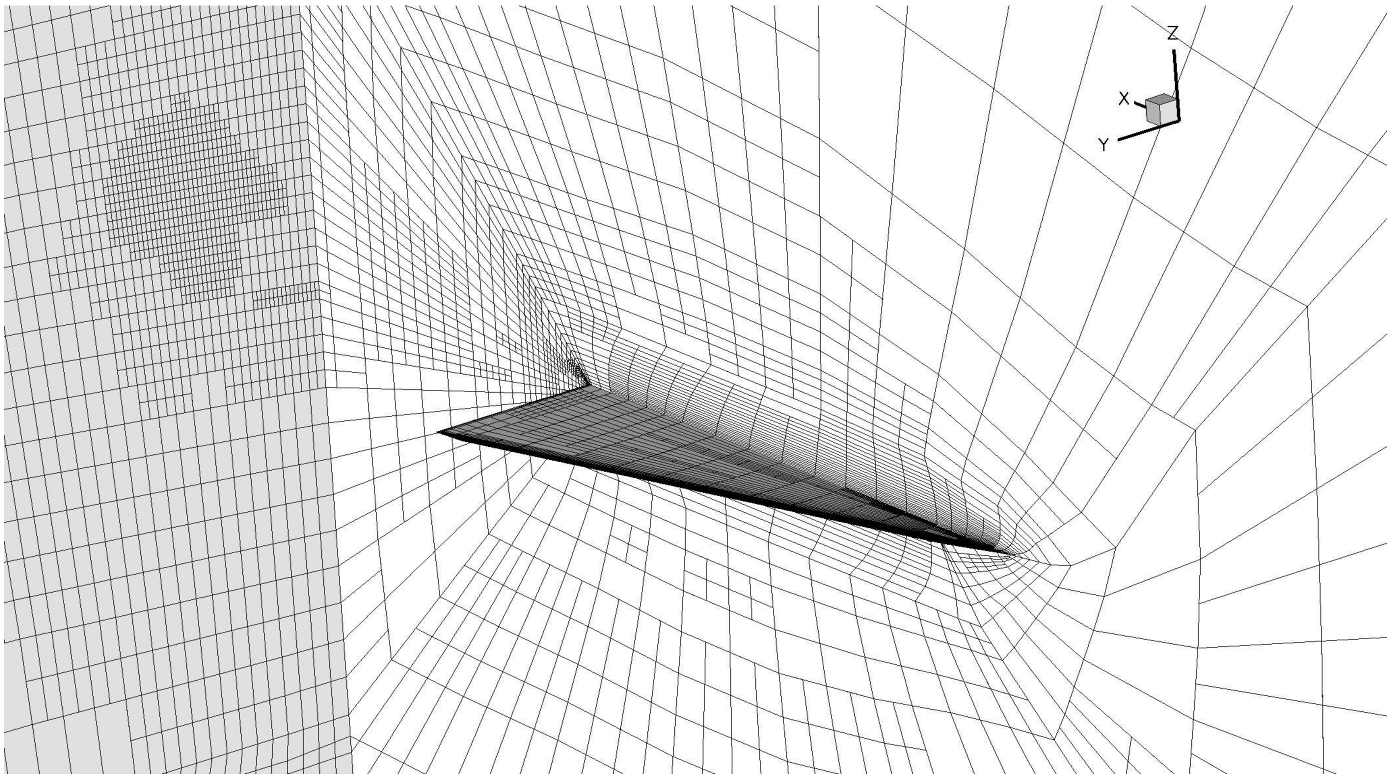


3. refinement step: 55 673 elements, 2 226 920 DoFs

Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3$, $\alpha = 12.5°$, $Re = 4000$.

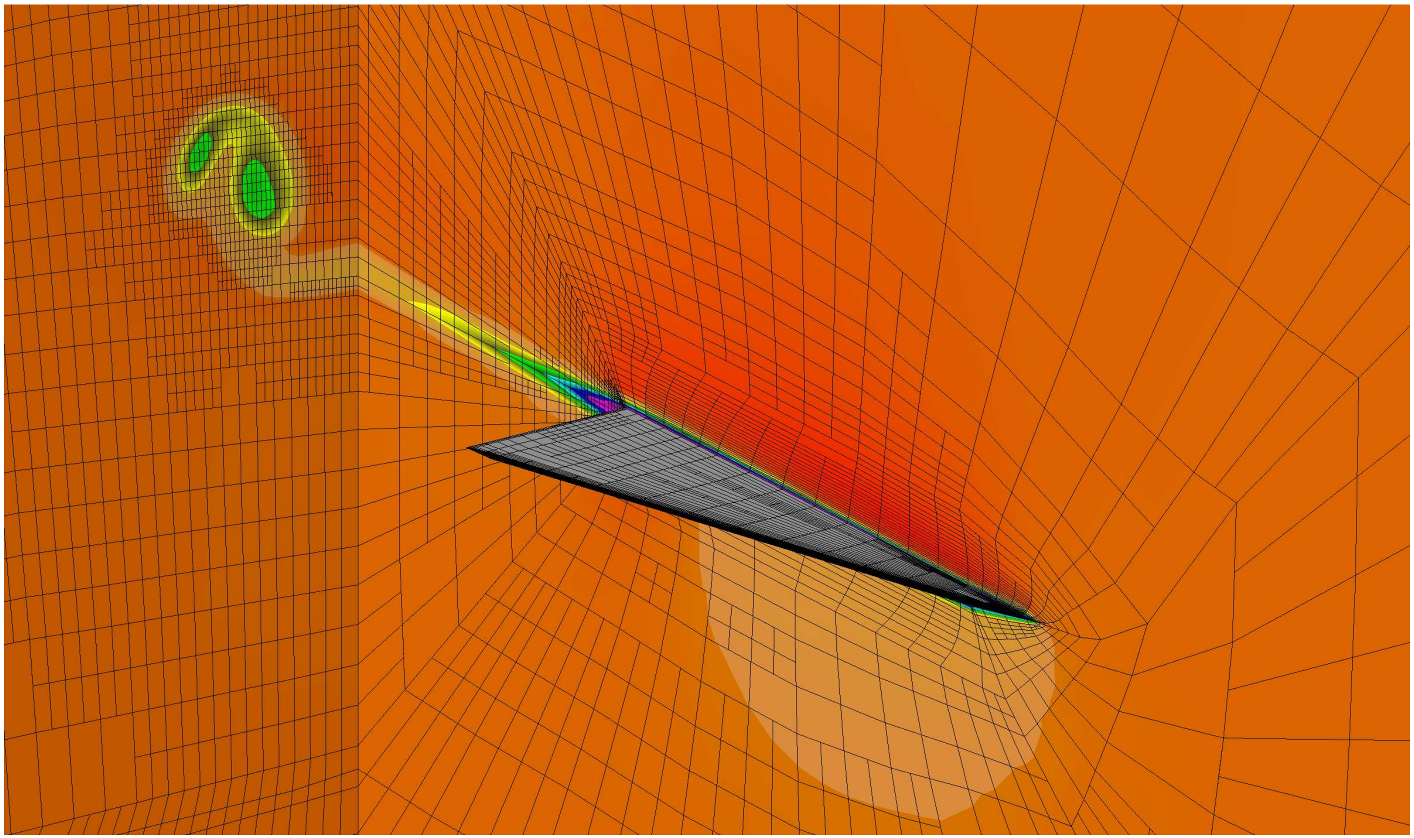
Residual-based refinement



4. refinement step: 144 279 elements, 5 771 160 DoFs

Residual-based refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3$, $\alpha = 12.5^\circ$, $Re = 4000$.



Error estimation and goal-oriented (adjoint-based) refinement

ADIGMA BTC3 test case: laminar flow around a delta wing.

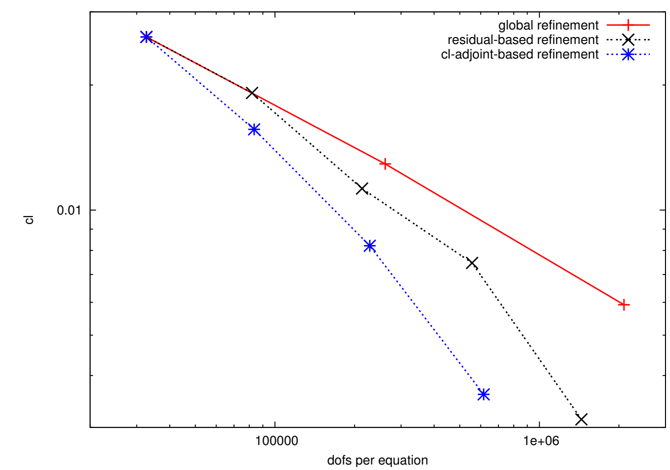
Freestream conditions: $M = 0.3$, $\alpha = 12.5°$, $Re = 4000$.

Reference values: $c_{\mathrm{l}}^{ref} = 0.3494$, $c_{\mathrm{d}}^{ref} = 0.1664$, $c_{\mathrm{m}}^{ref} = -0.0311$

$$J(\mathbf{u}) - J(\mathbf{u}_h) = \mathcal{R}(\mathbf{u}_h, \mathbf{z}) \approx \mathcal{R}(\mathbf{u}_h, \tilde{\mathbf{z}}_h) = \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa,$$

		error in c_l		
cells	DoFs	exact	estimate	ratio
3 264	130 560	-2.611e-02	-2.030e-02	0.78
8 346	333 840	-1.564e-02	-1.266e-02	0.81
22 843	913 720	-8.209e-03	-8.959e-03	1.09
61 567	2 462 680	-3.603e-03	-3.612e-03	1.00

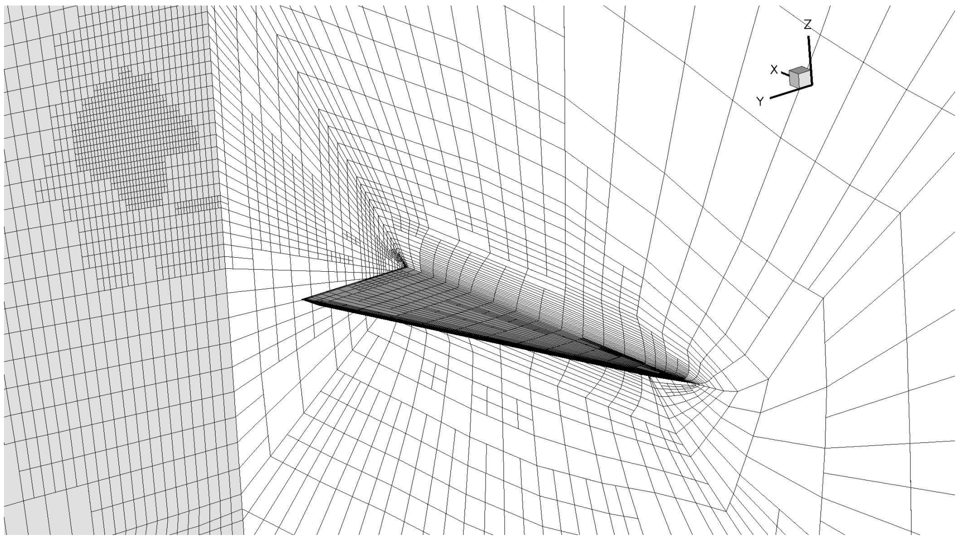
Error in c_l



Similar for c_d and c_m.

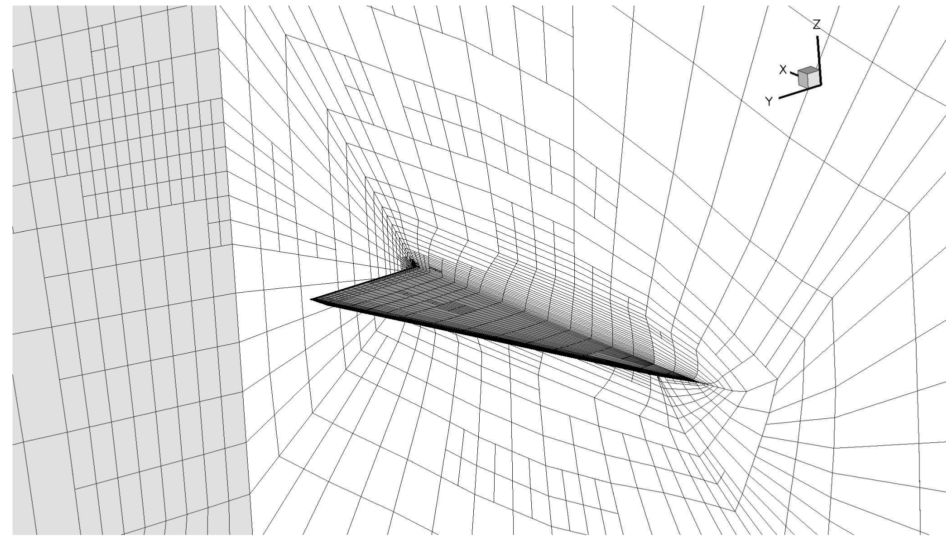
Local refinement for the laminar delta wing (BTC3)

Freestream conditions: $M = 0.3$, $\alpha = 12.5°$, $Re = 4000$.



residual-based refinement

5 771 160 DoFs, c_l: |error|=3.2e-03



adjoint-based refinement

2 462 680 DoFs, c_l: |error|=3.6e-03

Summary

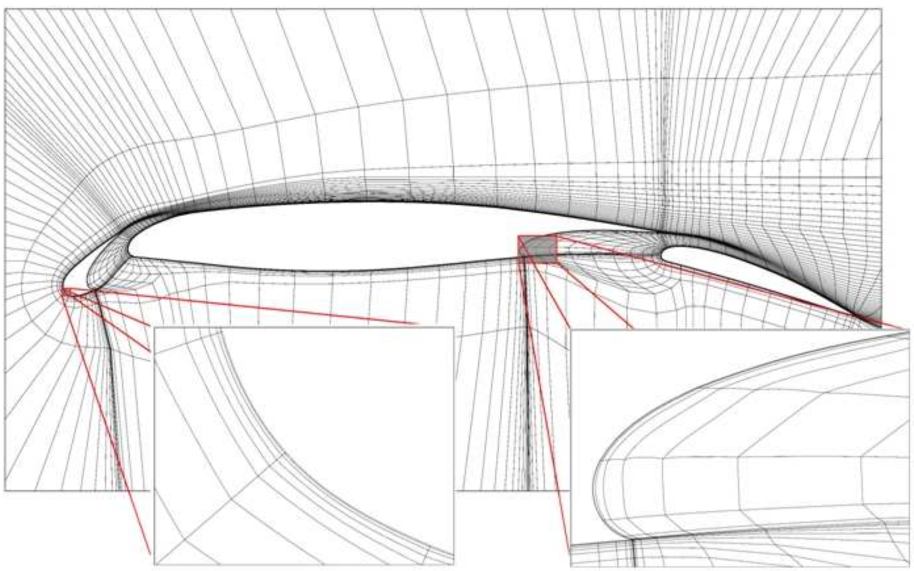
2d and 3d laminar compressible flows

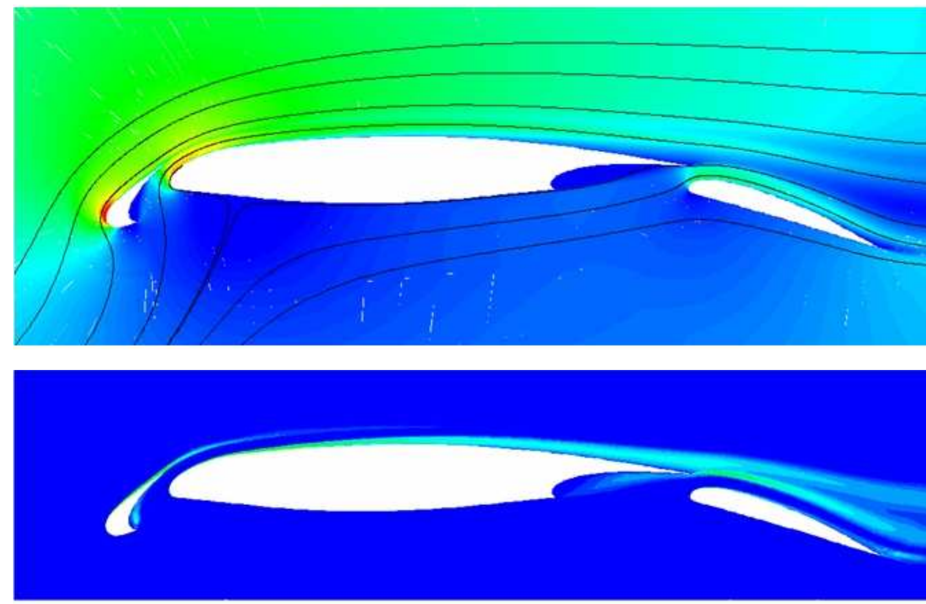
- **Error estimation and goal-oriented (adjoint-based) refinement for single and for multiple force coefficient**
- **Residual-based mesh refinement for 3d laminar flows**
- **Error estimation and goal-oriented mesh refinement for 3d laminar flows**

Outlook

Extension to turbulent flows

Example: L1T2 three element airfoil (high lift configuration)
$M = 0.197, \alpha = 20.18^\circ, Re = 3.52 \cdot 10^6$, **4th order DG discretization**





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- Florian Prill

Thank you.